Journal of Statistical Physics, Vol. 92, Nos. 1/2, 1998

Percolation and Minimal Spanning Trees

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Received August 15, 1997

Consider a random set $\mathcal{V}_n$ of points in the box $[n, -n)^d$, generated either by a Poisson process with density p or by a site percolation process with parameter p. We analyze the empirical distribution function F_n of the lengths of edges in a minimal (Euclidean) spanning tree $\mathcal{T}_n$ on $\mathcal{V}_n$. We express the limit of F_n, as $n \to \infty$, in terms of the free energies of a family of percolation processes derived from $\mathcal{V}_n$ by declaring two points to be adjacent whenever they are closer than a prescribed distance. By exploring the singularities of such free energies, we show that the large-n limits of the moments of F_n are infinitely differentiable functions of p except possibly at values belonging to a certain infinite sequence $(p_c(k): k \geq 1)$ of critical percolation probabilities. It is believed that, in two dimensions, these limiting moments are twice differentiable at these singular values, but not thrice differentiable. This analysis provides a rigorous framework for the numerical experimentation of Dussert, Rasigni, Rasigni, Palmari, and Llebaria, who have proposed novel Monte Carlo methods for estimating the numerical values of critical percolation probabilities.

KEY WORDS: Percolation; minimal spanning tree; free energy; critical value.

1. INTRODUCTION

Let $\mathcal{T}$ be the minimal (Euclidean) spanning tree on a finite subset S of $\mathbb{R}^d$. The geometry of $\mathcal{T}$ is a central subject in the theory and applications of combinatorial optimization. If S is chosen randomly from $\mathbb{R}^d$, then the probability distribution of $\mathcal{T}$ contains information about the "typical" structure of minimal spanning trees. In this paper, we study the geometry of $\mathcal{T}$ when S is chosen either according to product measure on the vertices

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of a lattice $\mathscr{L}$ or according to a Poisson process on $\mathbb{R}^d$. The purpose of our study is to develop the link between minimal spanning trees and percolation theory. This will be achieved by an analysis of the empirical distribution function of the edge-lengths of $\mathscr{T}$, and by relating this function to the "number of clusters per site" (or "free energy") of an appropriate site percolation model (in the lattice case, we shall have to consider an infinite family of site percolation models constructed on the vertex set of $\mathscr{L}$).

As mentioned above, the theory of random minimal spanning trees is quite well developed already; see refs. 2–4, 18, 22, 23, 25 for example. A relationship with percolation theory was proposed recently in refs. 8, 9, 16, and we shall explore this relationship further here. In refs. 8, 16 were reported the results of Monte Carlo experiments designed to estimate numerical constants (particularly the critical probabilities) associated with certain percolation models. It was proposed there that one may learn the value of a critical point by studying the locations of singularities of functions obtained as the limits of the sample mean and variance of the edge-lengths of a certain sequence $\mathscr{T}_n$ of minimal spanning trees defined on finite boxes in $\mathbb{R}^d$. The methods used in the current work are analytic and mathematically rigorous, and our conclusions are partially complementary to the propositions of refs. 8, 9.

We recall the definition of a minimal spanning tree. Let S be a finite subset of $\mathbb{R}^d$ where $d \geqslant 2$. A minimal spanning tree (MST) on S is a connected graph $\mathscr{T}$ having vertex set S such that the sum of the (Euclidean) edge-lengths of $\mathscr{T}$ is minimal. That is to say, we require that

$$\sum_{e \in \mathscr{T}} \|e\| = \min_G \sum_{e \in G} \|e\|$$

where $\|e\| = \|x - y\|$ is the Euclidean distance separating the endvertices x and y of the edge e, and the minimum is over all connected graphs G on the vertex set S.

Consider a realisation ω of a site percolation process, with density p, on a lattice $\mathscr{L}$ embedded in $\mathbb{R}^d$; note that ω is a random subset of the vertex set of $\mathscr{L}$. Write $\mathscr{T}_n$ for the MST on the vertex set $\omega \cap A_n$, where $A_n = [-n, n)^d$ is a box of side-length $2n$. Whereas refs. 8, 16 are directed largely at the sample mean and standard deviation of the edge-lengths of $\mathscr{T}_n$, we consider here the entire empirical distribution function of the edge-lengths. Writing F_n for this function (i.e., $F_n(\alpha)$ is the proportion of edges in $\mathscr{T}_n$ having length not exceeding α), we shall prove the convergence, as $n \to \infty$, of F_n to a certain deterministic limit function H_p. Furthermore $H_p(\alpha)$ may be expressed in terms of the "number of clusters per site" $\kappa_p(\alpha)$ defined as follows. For $\alpha \in (0, \infty)$ we construct a graph on ω by joining two points if and only if they are separated by a Euclidean distance not exceeding α. For $x \in \omega$, let $C_x^\alpha(\omega)$ denote the connected component (or "cluster") of this (infinite) graph containing the vertex x. Finally, we define $\kappa_p(\alpha) = \mathbb{E}_p(|C_0^\alpha|^{-1} \mid 0 \in \omega)$ where $\mathbb{E}_p$ denotes expectation. We shall prove that

$$\lim_{n \to \infty} F_n(\alpha) = 1 - \kappa_p(\alpha) \qquad \text{a.s. and in } \mathrm{L}^1$$

The last fact is actually rather straightforward (see Section 3). By exploring the weak convergence of the sequence $F_n(\cdot)$ of functions, we shall obtain also the convergence of the moments of $F_n(\cdot)$ to those of the limit function.

We turn now to the "singularities" proposed in the numerical work of ref. 8. The existence of such singularities is related to singularities of the family $\kappa_p(\alpha)$ of functions of p, as α ranges over the set $(0, \infty)$. The rigorous theory of such functions is incomplete (see ref. 11, Chap. 4), but it is believed that, for fixed α, the function $\kappa_p(\alpha)$ is a real-analytic function of p on $[0, 1]$ except at a certain critical value $p_c(\alpha)$. At the value $p_c(\alpha)$ in two dimensions, the function $\kappa_p(\alpha)$ is believed to be twice differentiable but not thrice differentiable. It is an open problem to prove that $\kappa_p(\alpha)$ is not infinitely differentiable on the entire interval $[0, 1]$. If we accept the physical picture just described, then we may deduce the existence of an infinite family of singularities of the limiting distribution function $H_p(\cdot) = 1 - \kappa_p(\cdot)$, i.e., an infinite set of values of p at which the moments of H_p are not infinitely differentiable functions of p. See Theorem (LP) (3.10) and Theorem (L) (3.11).

In the limit as $p \downarrow 0$, the percolation model converges weakly (when correctly re-scaled) to a Poisson process. Versions of the above statements are valid in the Poisson setting also, and we include these in the subsequent sections. For the basic properties of percolation and Poisson processes, we refer the reader to refs. 11 and 6, 19 respectively.

2. PRELIMINARIES

The meaning of the symbol $|A|$ will vary according to context: if A is a countable set, $|A|$ will denote the cardinality of A, and otherwise $|A|$ will denote the d-dimensional Lebesgue measure of A ($\subseteq \mathbb{R}^d$). Let $d \geqslant 2$. On $\mathbb{R}^d$, we denote the Euclidean norm by $\|\cdot\|$. If $A, B \subseteq \mathbb{R}^d$, we define $\|A, B\| = \inf\{\|a - b\| : a \in A, b \in B\}$.

We write $B_d(x, r) = B(x, r)$ for the closed Euclidean ball in $\mathbb{R}^d$ with radius r centred at x, and $\partial B_d(x, r) = \partial B(x, r)$ for its boundary. Let $B(r) = B(r, 0)$ and $\partial B(r) = \partial B(r, 0)$.

Suppose Γ is a finite weighted graph and $\alpha \in \mathbb{R}$ (see the appendix for graph-theoretic terminology). We denote by Γ^α the spanning subgraph of Γ obtained by deleting all edges of Γ whose weight strictly exceeds α. Let $E(\Gamma)$ be the number of edges of Γ. We define the "edge-weight distribution function" F_Γ of Γ by, for $\alpha \in \mathbb{R}$,

$$F_\Gamma(\alpha) = \begin{cases} \dfrac{E(\Gamma^\alpha)}{E(\Gamma)} & \text{if} \quad E(\Gamma) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

Let Ω be the collection of locally finite subsets of $\mathbb{R}^d$. We shall work on the box $A_n = [-n, n)^d$. For $\omega \in \Omega$, we write

$$\mathscr{V}_n(\omega) = \omega \cap A_n \qquad \text{and} \qquad V_n(\omega) = |\mathscr{V}_n(\omega)|$$

Let $\Gamma_n(\omega)$ be the complete graph on the finite set $\mathscr{V}_n(\omega)$. For each edge $\{x, y\}$ of $\Gamma_n(\omega)$, we assign to it the weight $\|x - y\|$. Let $\mathscr{T}_n(\omega)$ be any minimal spanning tree (MST) of the weighted graph $\Gamma_n(\omega)$. We write

$$F_n(\cdot, \omega) = F_{\mathscr{T}_n(\omega)}(\cdot) \tag{2.2}$$

By Corollary (A.4) in the appendix, $E(\mathscr{T}_n^\alpha(\omega))$ and F_n do not depend on the choice of the MST $\mathscr{T}_n$, but only on the set $\mathscr{V}_n$.

Suppose $\omega \in \Omega$. Let $\Gamma^\alpha(\omega)$ be the graph having vertex set ω and edge set consisting of all pairs $\{x, y\}$ with $x, y \in \omega$ and $0 < \|x - y\| \leqslant \alpha$. For $x \in \omega$, let $C_x^\alpha(\omega)$ be the collection of vertices in the connected component of $\Gamma^\alpha(\omega)$ containing the point x. We refer to $C_x^\alpha(\omega)$ as "the cluster at x when the maximal edge-length is set to α".

We interpret the term "lattice" $\mathscr{L}$ to mean a subset of $\mathbb{R}^d$ satisfying the following five conditions:

(i) $\mathscr{L}$ is locally finite,

(ii) $\mathscr{L}$ is invariant under translation by a unit vector in any of the co-ordinate directions,

(iii) $\mathscr{L}$ is invariant under permutations of the coordinates of $\mathbb{R}^d$, and also under the reflection $(x_1, x_2, ..., x_d) \mapsto (-x_1, x_2, ..., x_d)$,

(iv) for every pair $x, y \in \mathscr{L}$, there exists a translation τ and a rotation ρ of $\mathbb{R}^d$ such that $y = \tau(x)$ and $\mathscr{L} = \tau\rho(\mathscr{L})$,

(v) the origin 0 belongs to $\mathscr{L}$.

Note that a lattice is a set of points rather than a graph. The arguments and results of this paper are valid when applied to sets $\mathscr{L}$ of greater generality than required by (i)–(v), but we assume these five conditions here for simplicity of presentation.

For a given lattice $\mathscr{L}$ in $\mathbb{R}^d$, and $\alpha > 0$, we construct a graph $\mathscr{L}^\alpha$ by joining any pair of points in $\mathscr{L}$ which are separated by distance α or less. The ensuing graph will generally be disconnected, if α is small. There may exist ranges of values of α for which $\mathscr{L}^\alpha$ contains infinite components with dimensions strictly less than d. We shall concentrate on the component of $\mathscr{L}^\alpha$ containing the origin, which we denote by $\mathscr{L}^\alpha(0)$.

We now concentrate on two probability distributions for ω. Let $\mathscr{L}$ be a lattice in $\mathbb{R}^d$. The set ω will be distributed either as the set of open sites in a site percolation process with density p on $\mathscr{L}$ (we refer to this as "the lattice model"), or as a Poisson point process in $\mathbb{R}^d$ with intensity measure given by $p\,|\cdot|$ (we call this "the Poisson model"). When we refer to "both models" or "either model", then it is these two models that are meant. In both models we refer to p as the "density". The symbols $\mathbb{P}_p$ and $\mathbb{E}_p$ denote (respectively) probability and expectation with respect to either of these measures, unless we state explicitly that we are considering only one of the models. Because the statements of many of our results are identical for the two models, we usually use the same notation. When necessary, we use the superscripts L and P for the lattice and Poisson models, respectively; in addition, we write (L) (resp. (P)) in the title of any theorem or lemma which applies to the lattice (resp. Poisson) model, and (LP) when pertaining to both cases. For recent work on minimal spanning trees for the Poisson model, see refs. 4, 18, 22 and the references therein.

3. RESULTS

There are two (related) functions which are central to the analysis which follows, namely the quantities given by

$$(3.1) \qquad \kappa_p(\alpha) = \mathbb{E}_p\left(\frac{1}{|C_0^\alpha(\omega)|}\,\middle|\, 0 \in \omega\right), \qquad H_p(\alpha) = 1 - \kappa_p(\alpha)$$

In the case of the Poisson model, the above conditional expectation is interpreted in the usual way (see ref. 6). The quantity $\kappa_p(\alpha)$, viewed as a function of p with α held fixed, is often referred to in the physics literature as a "free energy" function. However, we shall refer here to $\kappa_p(\alpha)$ as the "number of clusters per site"; see Chapter 4 of ref. 11 for the basic properties of $\kappa_p(\alpha)$ in the lattice case. It is evident from the usual re-scaling argument that

$$(3.2) \qquad \kappa_p^{\mathrm{P}}(\alpha) = \kappa_1^{\mathrm{P}}(p^{1/d}\alpha)$$

in the case of the Poisson model; see refs. 15, 21 for more details of this case.

The following theorem contains the basic ingredients of the approach of this paper. The proof is straightforward, and a sketch thereof is presented at the end of this section.

(3.3) Theorem (LP). Let α, $p>0$. We have that, as $n\to\infty$,

$$F_n(\alpha, \omega)\to H_p(\alpha) \qquad \text{a.s. and in } \mathrm{L}^1$$

We shall show that H_p is a distribution function, and shall study its properties in some detail. Also, we shall investigate the convergence, as $n\to\infty$, of the sample moments of the distribution function F_n to those of H_p. Prior to doing this, we define a certain sequence of values of p at which difficulties arise. These are exactly the critical probabilities of a certain family of percolation models.

Let $\alpha>0$, and define the function

$$\theta^\alpha(p)=\mathbb{P}_p(|C_0^\alpha|=\infty \mid 0\in\omega)$$

Note that, in the lattice case, $\theta^\alpha(p)$ is essentially the percolation probability of the component $\mathscr{L}^\alpha(0)$ of the graph $\mathscr{L}^\alpha$. Using standard arguments from percolation theory (see refs. 11, 21), we have that there exists $p_c(\alpha)>0$ such that

$$\theta^\alpha(p)\begin{cases} =0 & \text{if } \; p<p_c(\alpha) \\ >0 & \text{if } \; p>p_c(\alpha)\end{cases}$$

For the Poisson model, the quantities $p_c(\alpha)=p_c^{\mathrm{P}}(\alpha)$ may be expressed in terms of one another: using the usual re-scaling, we have that $p_c^{\mathrm{P}}(\alpha)=\alpha^{-d}p_c^{\mathrm{P}}(1)$. The situation is more interesting in the lattice model. Suppose that we are working on the lattice $\mathscr{L}$, and let $\mathscr{D}(\mathscr{L})=\{\|x-y\|: x, y\in\mathscr{L}, x\neq y\}$ be the set of "inter-point" distances. It is easily seen that the set $\mathscr{D}(\mathscr{L})$ is countable, does not contain 0, and has no finite limit points. Consequently we may express $\mathscr{D}(\mathscr{L})$ in the form $\mathscr{D}(\mathscr{L})=\{\alpha_k: k\geqslant 1\}$ where

$$(3.4)\qquad 0<\alpha_k<\alpha_{k+1} \qquad \text{for} \quad k\geqslant 1, \qquad \text{and} \qquad \alpha_k\to\infty \qquad \text{as} \quad k\to\infty$$

It is clear that, for $k\geqslant 1$,

$$p_c(\alpha)=p_c(\alpha_k) \qquad \text{if} \quad \alpha_k\leqslant\alpha<\alpha_{k+1}$$

Here are some properties of the α_k and $p_c(\alpha_k)$. The proof is presented at the end of this section.

(3.5) Theorem (L). (i) The growth function of $\mathscr{L}$ satisfies $|\mathscr{L} \cap B(m)|/|B(m)| \to c_1$ as $m \to \infty$, for some constant $c_1 = c_1(\mathscr{L}) > 0$.

(ii) There exists a constant $c_2 = c_2(\mathscr{L}) > 0$ such that $\alpha_{k+1} \leqslant \alpha_k + 1$ and $c_2 k^{1/d} \leqslant \alpha_k \leqslant k$, for $k \geqslant 1$.

(iii) We have that $p_c(\alpha_k) \geqslant p_c(\alpha_{k+1})$ for all k, and $p_c(\alpha_k) \to 0$ as $k \to \infty$.

(iv) Let $K = \inf\{k: p_c(\alpha_k) < 1\}$. Then $(p_c(\alpha_k): k \geqslant K)$ is a strictly decreasing sequence.

In order to study the properties of the empirical edge-length distribution function $F_n(\cdot, \omega)$, we shall need certain information about the smoothness of the limit function $H_p(\cdot)$. The necessary facts are contained in the next two theorems.

(3.6) Theorem (LP). Let $\alpha \in (0, \infty)$. Viewed as a function of p, $\kappa_p(\alpha)$ is continuously differentiable on its domain. (This includes the statement that $\kappa_p(\alpha)$ has one-sided derivatives at all finite boundary points.) It is infinitely differentiable (with one-sided derivatives where appropriate) except possibly at the critical point $p = p_c(\alpha)$.

Standard physics dogma asserts that, in two dimensions, $\kappa_p(\alpha)$ is twice differentiable at $p_c(\alpha)$ but not thrice differentiable, but no proof is known of this statement (see ref. 11, p. 78).

(3.7) Theorem (LP). If $p > 0$, then $H_p(\cdot)$ is a distribution function. In the Poisson case, the measure corresponding to $H_p = H_p^{\mathrm{P}}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, and there exist strictly positive constants γ_0 and γ_1 such that

$$\frac{\partial H_p^{\mathrm{P}}(\alpha)}{\partial \alpha} \leqslant \gamma_0 \alpha^{d-1} \exp(-\gamma_1 p \alpha^d) \qquad \text{for} \quad \alpha > 0 \tag{3.8}$$

We turn now to the definition of the moments of the distribution functions H_p and F_n, namely the quantities given when $p > 0$ by

$$m_j(p) = \int_{[0,\infty)} \alpha^j \, dH_p(\alpha), \qquad M_{j,n}(\omega) = \int_{[0,\infty)} \alpha^j \, dF_n(\alpha, \omega) \tag{3.9}$$

(3.10) Theorem (LP). Let $j \geqslant 1$.

(i) We have that $M_{j,n}(\omega) \to m_j(p)$ a.s. and in L^1 as $n \to \infty$.

(ii) The function $m_j(p)$ is differentiable except at the point $p = 0$. It is infinitely differentiable in the lattice model except possibly when $p \in \{p_c(\alpha_k): k \geqslant 1\}$. In the Poisson model, it is the case that $m_j^{\mathrm{P}}(p) = p^{-j/d} m_j^{\mathrm{P}}(1)$ for $p > 0$.

(iii) We have that $p^{j/d} m_j(p) \to m_j^{\mathrm{P}}(1)$ as $p \to 0$, where $m_j^{\mathrm{P}}(1)$ is given as in (3.9) with H_1 replaced by H_1^{P}.

As remarked above, for a given percolation model, it is believed that the number of clusters per site (or "free energy") is not infinitely differentiable at the critical point. Such a singularity would be shared by the moments $m_j(p)$ given above. We state this formally in the following way.

(3.11) Theorem (L). Let $k, j, r \geqslant 1$, and consider the lattice model. If $m_j(p)$ is r times differentiable at the point $p_c(\alpha_k)$ (<1), then $\kappa_p(\alpha_k)$ is r times differentiable as a function of p at this point.

In ref. 8 is proposed the following method for estimating critical probabilities. Consider a realisation of a site percolation process on a graph (such as the square or triangular lattice). Construct a MST on the points within the box A_n and consider the empirical edge-length distribution function F_n. After a sequence of numerical experiments, one may plot the mean and variance of F_n against the variable p. The experiments of ref. 8 suggest that these functions have singularities at the critical point of the lattice, and it was proposed that the above procedure might be used in order to estimate the true value of this critical point. We make two remarks about this procedure. First, the results of the current paper indicate that these functions have an infinite set of singularities. Secondly, the outcome of the numerical procedure depends on the actual embedding of the lattice in $\mathbb{R}^d$.

In advance of delivering proofs of Theorem (LP) (3.3) and Theorem (L) (3.5), we summarise the contents of the remainder of the paper. Throughout, we present proofs only when novelty is required. Some of the required arguments are fairly standard, in which cases references are given and the details omitted. Section 4 contains some technical estimates of use later. Theorem (LP) (3.6) is proved in Section 5, and Theorem (LP) (3.7) in Section 6. The remaining two sections contain the proofs of Theorem (LP) (3.10) and Theorem (L) (3.11). Graph-theoretic definitions have been relegated to the appendix.

Sketch Proof of Theorem (LP) (3.3). It follows from elementary graph-theoretic considerations (see the appendix) that

$$F_n(\alpha, \omega) = \frac{V_n(\omega) - K_n(\alpha, \omega)}{V_n(\omega) - 1} \quad \text{if} \quad V_n(\omega) \geqslant 2 \tag{3.12}$$

where $K_n(\alpha, \omega)$ is the number of components of the graph $\Gamma_n^\alpha(\omega)$. We summarise the basic steps of the proof of the theorem in the case of the lattice model; for the details see page 73 of ref. 11, where they are presented for nearest-neighbour bond percolation on the cubic lattice. We have that, as $n \to \infty$,

$$\frac{V_n(\omega) - 1}{|A_n \cap \mathscr{L}|} \to p \qquad \text{a.s. and in } \mathrm{L}^1$$

by the strong law of large numbers. Use of the ergodic theorem yields that

$$\lim_{n \to \infty} \frac{K_n(\alpha, \omega)}{|A_n \cap \mathscr{L}|} = \mathbb{E}_p\left(\frac{1\{0 \in \omega\}}{|C_0^\alpha(\omega)|}\right) \qquad \text{a.s. and in } \mathrm{L}^1$$

where $1\{A\}$ is the indicator function of the event A. The a.s. convergence in the statement of the theorem follows immediately. Convergence in L^1 then follows by use of the bounded convergence theorem.

In the Poisson case, the claim of the theorem may be obtained by approximating to $\mathbb{R}^d$ with a discrete grid, and by taking the limit as the grid size approaches 0. ∎

Proof of Theorem (L) (3.5). (i) This follows from the local finiteness and translation invariance of $\mathscr{L}$.

(ii) Since 0 lies in $\mathscr{L}$, and $\mathscr{L}$ is invariant under unit shifts, we have that $\mathbb{Z}^d \subseteq \mathscr{L}$. It follows that $\mathscr{D}(\mathscr{L}) \supseteq \{1, 2, ...\}$, whence $\alpha_k \leqslant k$ and $\alpha_{k+1} \leqslant \alpha_k + 1$. There exists a strictly positive constant $b = b(\mathscr{L})$ such that the number of points y of $\mathscr{L}$ satisfying $\|y\| \leqslant r$ does not exceed br^d. Let $k \geqslant 1$. There exist $x, y \in \mathscr{L}$ such that $\|x\| \leqslant \sqrt{d}$ and $\|x - y\| = \alpha_k$. By the triangle inequality, $\|y\| \leqslant \alpha_k + \sqrt{d}$, and therefore

$$k \leqslant b(\alpha_k + \sqrt{d})^d \cdot b(\sqrt{d})^d$$

since α_k is a non-decreasing sequence. The lower bound on α_k follows.

(iii) The monotonicity of $p_c(\alpha_k)$ is trivial since $\alpha_k \leqslant \alpha_{k+1}$. We use a simple block argument for the second part. Let $p > 0$ and choose an integer M such that

$$\mathbb{P}_p(A_M \text{ contains a point of } \omega) > p_c(\mathbb{Z}^2) \tag{3.13}$$

where $p_c(\mathbb{Z}^2)$ is the critical probability of site percolation on the square lattice. Now choose k such that $\alpha_k > 4M\sqrt{d}$. With this choice of k, we have that $\|x-y\| < \alpha_k$ for all pairs x, y satisfying $x \in A_M$, $y \in A_M + (2M, 0, 0, \ldots, 0)$. For $x_1, x_2 \in \mathbb{Z}$, we colour the translate $A_M + 2M(x_1, x_2, 0, \ldots, 0)$ of A_M *green* if it contains some point of ω. The green boxes constitute a site percolation process on a copy of the square lattice, each site of which represents the corresponding translate of A_M. By (3.13), this process is supercritical. It follows that $p_c(\alpha_k) \leqslant p$.

(iv) A detailed argument would be long and contain no new ideas, and is therefore omitted. The claim may be shown by following the methods of ref. 1. ∎

4. TECHNICALITIES

We establish next certain inequalities which will be necessary for the proofs which follow later.

4.1. Re-scaling

For $r>0$ and $x \in \mathbb{Z}^d$, define the box $A_r^x = 2rx + (r, r, \ldots, r) + [-r, r)^d$. For fixed r, the A_r^x will serve as sites in a re-scaled process. Suppose $r>0$ and $\omega \in \Omega$. We call x $(\in \mathbb{Z}^d)$ *r-occupied* if $\omega \cap A_r^x \neq \varnothing$, and we write $\Theta_r\omega$ for the set of r-occupied sites of $\mathbb{Z}^d$. If ω is a realisation of any of the models considered in this paper, $\Theta_r\omega$ is then a realisation of a site percolation process on $\mathbb{Z}^d$. So long as r is chosen suitably, we have that $p(x) = \mathbb{P}_p$ (x is r-occupied) does not depend on the choice of $x \in \mathbb{Z}^d$. Note that

$$p(x) = \begin{cases} 1 - e^{-p(2r)^d} & \text{for the Poisson model, with } r>0, \\ 1-(1-p)^{(2r)^d} & \text{for the lattice model on } \mathbb{Z}^d, \text{ with } r = 1, 2, \ldots \end{cases}$$

(4.1.1) Lemma (LP). If $\alpha \in \mathbb{R}$, $\omega \in \Omega$, and $r>0$, then

$$1 - F_n(\alpha, \omega) \leqslant 1 - F_{n/(2r)}\left(\frac{\alpha - 2r\sqrt{d}}{2r}, \Theta_r\omega\right)$$

for all positive multiples n of $2r$.

Proof. We use a fairly standard argument in order to build a spanning tree of $\omega \cap A_n$ by joining together minimal spanning trees on certain subsets of A_n. Let $\omega \in \Omega$, let $n, r > 0$, and let $m = n/(2r)$, assumed integral. We partition A_n into boxes, each of which is a translate of A_r, in such a way that the centres of these boxes form a copy of A_m (suitably re-scaled and translated). Let $\bar{\omega} = \Theta_r \omega$. We may assume that $V_m(\bar{\omega}) > 1$ and $\alpha \geqslant 2r\sqrt{d}$, and we do so for the rest of the proof. Let $\mathscr{T}_m(\bar{\omega})$ be a MST of $\bar{\omega} \cap A_m$ and, for each $x \in A_m$ which is r-occupied, let $\mathscr{T}_r(\omega, x)$ be a MST of $\omega \cap A_r^x$. We construct a spanning tree of $\omega \cap A_n$ as follows. First, take the union of all such $\mathscr{T}_r(\omega, x)$, noting that this union is a spanning forest of $\omega \cap A_n$. For all pairs $x, y \in \bar{\omega} \cap A_m$ which are joined by an edge of $\mathscr{T}_m(\bar{\omega})$, we add an edge joining some point (chosen according to a predetermined rule) in $\omega \cap A_r^x$ to some point in $\omega \cap A_r^y$. The resulting graph is a spanning tree of $\omega \cap A_n$. We claim that, by Corollary (A.4) in the Appendix,

$$(4.1.2) \quad F_n(\alpha, \omega) \geqslant \frac{1}{V_n(\omega) - 1} \left\{ \sum_{x \in A_m} E(\mathscr{T}_r(\omega, x)) + E_m\left(\frac{\alpha - 2r\sqrt{d}}{2r}, \bar{\omega}\right) \right\}$$

The following two facts are relevant to this inequality. First, the diameter of A_r is $2r\sqrt{d}$ ($\leqslant \alpha$), whence all edges in any given $\mathscr{T}_r(\omega, x)$ have length not exceeding α. Secondly, if there is an edge $\{x, y\}$ of $\mathscr{T}_m(\bar{\omega})$ with length not exceeding $(\alpha - 2r\sqrt{d})/(2r)$, then the above construction requires that an edge be added in A_n joining some point in A_r^x to some point in A_r^y; no two such points are further than α apart. Now,

$$\sum_{x \in A_m} E(\mathscr{T}_r(\omega, x)) = V_n(\omega) - V_m(\bar{\omega})$$

by (A.1). Substituting this into (4.1.2) yields the result. ∎

4.2. Comparison with Geometrically Distributed Random Variables

It will at times be useful to bound the edge-length distribution function of a random MST in terms of the empirical distribution function of a sequence of geometrically distributed random variables. We shall use such an inequality only for the case of the lattice model on $\mathbb{Z}^d$, and therefore we consider this case only in this subsection.

We say that a random variable Z is *geometric-p* if Z takes the value k with probability $(1-p)^{k-1}p$, for $k = 1, 2, \ldots$.

(4.2.1) Theorem (L). Let ω denote a realisation of a site percolation process on $\mathbb{Z}^d$ with density p, and let V_n and F_n be as in Section 2. There exists a sequence $X_0(\omega), X_1(\omega), \ldots$ of independent geometric-p random variables, depending on ω alone, such that

$$1 - F_n(\alpha, \omega) \leqslant 1 - \tilde{F}_n(\alpha, \omega) \qquad \text{for all} \quad \alpha \in \mathbb{R},$$

where $\tilde{F}_n(\cdot, \omega)$ denotes the empirical distribution function of the random sequence $X_1(\omega), ..., X_{V_n(\omega)-1}(\omega)$.

Proof. We turn $\mathbb{Z}^d$ into a graph by adding edges between any two points x, y satisfying $\|x-y\|=1$. Let $\omega \subseteq \mathbb{Z}^d$. Let π be a fixed nearest-neighbour path in $\mathbb{Z}^d$ that starts at the origin and visits each site in $\mathbb{Z}^d$ exactly once; we require that, for every $n \geqslant 0$, π visits every point in A_n before visiting any point in $A_{n+1} \setminus A_n$. Let $\pi(1), \pi(2), ...$ be the sites in π in the order visited. Let $i_1(\omega), i_2(\omega), ...$ be the increasing sequence of indices i for which $\pi(i) \in \omega$. For $k \geqslant 1$, let $\sigma_k(\omega) = \pi(i_k(\omega))$ and let $\eta_k(\omega)$ be the straight line segment in $\mathbb{R}^d$ joining $\sigma_k(\omega)$ to $\sigma_{k+1}(\omega)$. Let $\Sigma(\omega)$ be the path $\sigma_1(\omega), \eta_1(\omega), \sigma_2(\omega), \eta_2(\omega), ...$, and let $T_n(\omega)$ be the sub-path $\sigma_1(\omega), \eta_1(\omega), \sigma_2(\omega), \eta_2(\omega), ..., \sigma_{V_n}(\omega)$ (if $V_n(\omega) > 0$). Then $T_n(\omega)$ is a spanning tree of $\mathscr{V}_n(\omega)$. Let $X_0(\omega) = i_1(\omega)$ and, for $k \geqslant 1$, let $X_k(\omega) = i_{k+1}(\omega) - i_k(\omega)$ and $l_k(\omega) = \|\sigma_{k+1}(\omega) - \sigma_k(\omega)\|$. We have that $l_k(\omega) \leqslant X_k(\omega)$ for $k \geqslant 1$.

Suppose now that ω is distributed as the set of open sites of a percolation process on $\mathbb{Z}^d$ with density p. Then the X_k are independent geometric-p random variables. Assume $V_n(\omega) > 1$, and let $\tilde{F}_n(\omega)$ be the empirical distribution function of the sequence $X_1(\omega), X_2(\omega), ..., X_{V_n-1}(\omega)$. Let $F_{T_n(\omega)}$ be as in (2.1) with $\Gamma = T_n(\omega)$ and using the weight function $\|\cdot\|$. Then $F_{T_n(\omega)}$ is the empirical distribution function of the sequence $l_1(\omega), l_2(\omega), ..., l_{V_n-1}(\omega)$. Using Corollary (A.4), we deduce that

$$1 - F_n(\alpha, \omega) \leqslant 1 - F_{T_n(\omega)}(\alpha) \leqslant 1 - \tilde{F}_n(\alpha, \omega)$$

for all ω with $V_n(\omega) > 1$, and for all α. ∎

4.3. Super-Exponential Bounds for the Number of Edges

Our purpose here is to prove a bound on the tail of the number of long edges in a MST.

(4.3.1) Lemma (LP). There exist strictly positive constants K_0, γ_0, and γ_1 such that

$$\frac{\mathbb{E}_p(E(\mathscr{T}_n^\infty(\omega)) - E(\mathscr{T}_n^\alpha(\omega)))}{|A_n|} \leqslant \gamma_0 p \exp(-\gamma_1 p \alpha^d) \qquad \text{if} \quad \alpha \geqslant K_0$$

Proof. We go into some detail in the Poisson case. The proof in the lattice case is similar but some additional complications arise because of the lattice spacing; see the remarks at the end of the proof.

Let $0 < \alpha < \beta$. If there is an edge in $\mathscr{T}_n(\omega)$ whose length lies in the interval $(\alpha, \beta]$, then it must join two points x and y such that $\alpha < \|x-y\| \leqslant \beta$, and in addition (by examining Kruskal's greedy algorithm) there can exist no point of $\omega \cap A_n$ belonging to $B(x, \alpha) \cap B(y, \alpha)$. Therefore, for large n,

$$(4.3.2) \quad E_n(\beta) - E_n(\alpha) \leqslant \sum_{x \in \omega \cap A_n} \sum_{y \in \omega \cap A_n \cap A(x, \alpha, \beta)} 1\{B(x, \alpha) \cap B(y, \alpha) \cap \omega \cap A_n = \varnothing\}$$

where $E_n(\gamma) = E(\mathscr{T}_n^{\gamma}(\omega))$, $A(x, \alpha, \beta)$ is the annulus enclosed by the spheres centred at x and having radii α and β respectively, and $1\{\cdot\}$ denotes the indicator function.

An estimate of the expected value of (4.3.2) yields

$$(4.3.3) \qquad \frac{\mathbb{E}_p(E_n(\beta) - E_n(\alpha))}{|A_n|\,(\beta - \alpha)} \leqslant \gamma_0 \beta^{d-1} p^2 \exp(-\gamma_1 p(2\alpha - \beta)^d) \quad \text{if} \quad 0 < \tfrac{1}{2}\beta < \alpha < \beta$$

where γ_0 and γ_1 depend only on the dimension d. We have used the fact that, if $\frac{1}{2}\beta < \alpha < \|x - y\| \leqslant \beta$, then $B(x, \alpha) \cap B(y, \alpha)$ contains a Euclidean ball of diameter no smaller than $\alpha - (\|x - y\| - \alpha) \geqslant 2\alpha - \beta$, and at least half of this ball lies in A_n. To derive the bound in the statement of the lemma, take $\alpha = k - 1$ and $\beta = k$ in (4.3.3), and then sum over $k \geqslant K + 1$ where K is an integer. This gives that, for sufficiently large K,

$$\begin{aligned} \frac{\mathbb{E}_p(E_n(\infty) - E_n(K))}{|A_n|} &\leqslant \gamma_0 p^2 \sum_{k=K+1}^{\infty} k^{d-1} \exp(-p\gamma_2(k-2)^d) \\ &\leqslant \gamma_3 p^2 \int_K^{\infty} x^{d-1} \exp(-p\gamma_4 x^d)\, dx \\ &= \gamma_5 p \exp(-p\gamma_4 K^d) \end{aligned}$$

where the γ_i are positive and depend only on d. This, taken together with the fact that $E_n(\alpha)$ is increasing in α, gives the result, for newly defined γ_0, γ_1.

Here are some further notes for the lattice case. Suppose that

$$(4.3.4) \qquad \alpha_k \leqslant \alpha < \alpha_{k+1} \leqslant \alpha_l \leqslant \beta < \alpha_{l+1}$$

where $0 \leqslant k < l$ (and $\alpha_0 = 0$). If $x, y \in \mathscr{L}$ and $\alpha < \|x - y\| \leqslant \beta$, then $\|x - y\| \in \{\alpha_{k+1}, \ldots, \alpha_l\}$, and any point in $B(x, \alpha) \cap B(y, \alpha) \cap \mathscr{L}$ actually lies in $B(x, \alpha_k) \cap B(y, \alpha_k)$. This intersection contains a ball of diameter at least $2\alpha_k - \|x - y\| \geqslant 2\alpha_k - \alpha_l$. By Theorem (L) (3.5)(ii) and (4.3.4), this diameter is at least $2\alpha - 2 - \beta$. We argue now as in the Poisson case. Note that there exist constants $c_i = c_i(\mathscr{L})$ such that, for $x \in \mathscr{L}$,

$$\begin{aligned}|A(x, \alpha, \beta) \cap \mathscr{L}| &\leqslant |A(0, \alpha - \sqrt{d}, \beta + \sqrt{d}) \cap \mathscr{L}| \\ &\leqslant c_1 \, |A(0, \alpha - \sqrt{d}, \beta + \sqrt{d}) \cap \mathbb{Z}^d| \\ &\leqslant c_2(\beta + \sqrt{d})^{d-1} \, [(\beta + \sqrt{d}) - (\alpha - \sqrt{d})]\end{aligned}$$

We have therefore that there exist positive constants c_4, n_0 such that

$$\frac{\mathbb{E}_p(E_n(\beta) - E_n(\alpha))}{|A_n| \, (\beta - \alpha + 2\sqrt{d})} \leqslant c_3 \beta^{d-1} p^2 \exp(c_4(2\alpha - 2 - \beta)^d \log(1 - p))$$

if $\beta > \alpha > a_0$ and $2(\alpha - a_0) > 2 + \beta$. We note that $\log(1-p) \leqslant -p$ and proceed as before. ∎

4.4. Tail Estimates for Finite Percolation Clusters

Let us consider site percolation on the graph $\mathscr{L}^\alpha$, where $\alpha > 0$. We prove next two lemmas concerning the tail of the radius of the open cluster C_0^α at the origin. The first deals with the subcritical case ($p < p_c(\alpha)$), and the second with the supercritical case. We define the *radius* $\mathrm{rad}(S)$ of a subset S of $\mathbb{R}^d$ containing the origin by $\mathrm{rad}(S) = \inf\{r \in \mathbb{R} : S \subseteq A_r\}$.

(4.4.1) Lemma (L). Assume $d \geqslant 2$ and $\alpha > 0$. There exists a function $\nu = \nu(p, \alpha)$, strictly positive and continuous in p when $p < p_c(\alpha)$, such that

$$\mathbb{P}_p(\mathrm{rad}(C_0^\alpha) \geqslant n) \leqslant e^{-n\nu} \qquad \text{for} \quad n \geqslant 1$$

(4.4.2) Lemma (L). Assume $d \geqslant 2$, $\alpha > 0$, and $p_c(\alpha) < 1$.

(a) There exists a function $\xi = \xi(p, \alpha)$, strictly positive and continuous in p when $p > p_c(\alpha)$, such that

$$\mathbb{P}_p(n < \mathrm{rad}(C_0^\alpha) < \infty) \leqslant e^{-n\xi} \qquad \text{for} \quad n \geqslant 1$$

(b) Let $0 < p_0 < 1$. There exist positive constants a_0, γ, σ, such that, if $p_0 \leqslant p \leqslant 1$ and $\alpha \geqslant a_0$, then

$$\mathbb{P}_p(n < \mathrm{rad}(C_0^\alpha) < \infty) \leqslant \gamma(1 - p)^{n\sigma\alpha^{1 \vee (d-2)}} \qquad \text{for} \quad n \geqslant 1$$

where $a \vee b = \max\{a, b\}$.

There exists a positive constant $c = c(\mathscr{L})$ such that

$$\text{(4.4.3)} \qquad \text{if} \quad |C_0^\alpha| \geqslant n \qquad \text{then} \qquad \text{rad}(C_0^\alpha) \geqslant cn^{1/d}$$

Using this fact, one may obtain an estimate for the tail of the *volume* of an open cluster from an estimate for its radius. Estimates obtained from the above lemmas in this way are not the best possible, but will suffice for the purposes of this paper. They may be strengthened by utilising further methods of ref. 11.

Proof of Theorem (L) (4.4.1). We do not present this, since it follows the proof of Theorem (3.4) of ref. 11. ▮

Proof of Lemma (L) (4.4.2). Suppose that $d \geqslant 3$ (we shall return later to the case $d = 2$). First we prove part (b), and then we indicate the necessary steps in order to obtain part (a). We begin with two sub-lemmas.

(4.4.4) Lemma (L). For $d \geqslant 2$, there exist $p_1 \in (0, 1)$ and constants γ, $M > 0$, depending on d and p_1, such that

$$\mathbb{P}_p^{\mathbb{Z}^d}(|C_0^1| < \infty) \leqslant \gamma(1-p)^M \qquad \text{if} \quad p_1 \leqslant p \leqslant 1$$

where the superscript indicates that the lattice in question is $\mathbb{Z}^d$.

Proof. By an argument presented in ref. 11 (remarks on Theorem (6.95), pp. 138–140), there exist positive constants γ_0, μ, and ν, depending only on d, such that

$$\mathbb{P}_p^{\mathbb{Z}^d}(|C_0^1| < \infty) \leqslant 1 - p + \gamma_0 p \sum_{n=1}^{\infty} n^d[(1-p)\mu]^{\nu n^{(d-1)/d}} \qquad \text{if} \quad (1-p)\mu < 1$$

Therefore, with p_1 chosen to satisfy $(1 - p_1)\mu < 1$, there exists a constant $\gamma_1 = \gamma_1(\gamma_0, \mu, \nu, p_1)$ such that

$$\mathbb{P}_p^{\mathbb{Z}^d}(|C_0^1| < \infty) \leqslant (1-p) + \gamma_1(1-p)^\nu \qquad \text{if} \quad p_1 \leqslant p \leqslant 1 \quad ▮$$

(4.4.5) Lemma (L). Let $d \geqslant 2$ and $p_0 \in (0, 1)$. There exist constants a_0, γ, $N > 0$, depending on d and $\mathscr{L}$, such that

$$\mathbb{P}_p(|C_0^\alpha| < \infty) \leqslant \gamma(1-p)^{N\alpha^d} \qquad \text{if} \quad p_0 \leqslant p \leqslant 1 \quad \text{and} \quad \alpha \geqslant a_0$$

Proof. In the notation of Section 4.1,

$$\mathbb{P}_p(|C_0^\alpha(\omega)| < \infty) \leqslant \mathbb{P}_p(|C_0^1(\Theta_{r(\alpha)}\omega)| < \infty) = \mathbb{P}_\pi^{\mathbb{Z}^d}(|C_0^1| < \infty)$$

where $r(\alpha)=\lfloor \alpha/\sqrt{4d+12} \rfloor$ and $\pi=\pi(p,\alpha)=1-(1-p)^{f(\alpha)}$ with $f(\alpha)=|A_{r(\alpha)}\cap \mathscr{L}|$. Let p_1, γ, M be given as in Lemma (L) (4.4.4), and choose a_0 such that $f(a_0)\geqslant \log(1-p_1)/\log(1-p_0)$. Assume $p\geqslant p_0$, so that $\pi(p,\alpha)\geqslant p_1$, whence

$$\mathbb{P}_p(|C_0^\alpha|<\infty)\leqslant \gamma(1-\pi)^M=\gamma(1-p)^{Mf(\alpha)}\leqslant \gamma(1-p)^{N\alpha^d}$$

where N depends only on M and $\mathscr{L}$. ▮

We turn now to the proof of Lemma (L) (4.4.2)(b) proper, when $d\geqslant 3$ and $0<p_0<1$. We adapt an argument of ref. 5 (see ref. 11, pp. 127–129). For $r\in\mathbb{R}$, let $P_r=\{x\in\mathbb{R}^d : x_1=r\}$ and $Q_r=\{x\in\mathbb{R}^d : x_1\geqslant r\}$. For integral i, the set $\mathscr{L}\cap P_i$ constitutes a $(d-1)$-dimensional lattice, and we pick constants a_0, γ, N according to Lemma (L) (4.4.5) with d replaced by $d-1$. We now follow ref. 11 but working with $(d-1)$-dimensional hyperplanes rather than with slabs. Instead of reproducing the argument in full detail, we summarise the necessary changes. Let $\alpha\geqslant a_0\vee\sqrt{d}\vee\frac{3}{2}$, let $\gamma=\sup\{x_1 : x\in\mathscr{L},\ \|x\|\leqslant\alpha\}$, and $\beta=\lceil\gamma\rceil$; note that $\lfloor\alpha\rfloor\leqslant\gamma\leqslant\alpha$ and $\beta\leqslant 2\gamma$. Let $x\in\mathscr{L}$ be such that $x_1=\gamma$. As in ref. 11, we build the cluster C_0^α according to a recursive construction, and we write $v_i=(a_1,a_2,a_3,\ldots,a_d)$ for the earliest open vertex encountered which lies in the half-space $Q_{i\beta}$. There are two cases to be considered.

(i) If $a_1+\gamma<(i+1)\beta$, we define $w_i'=v_i+x$, and we choose $w_i''\in P_{(i+1)\beta}$ such that $\|w_i'-w_i''\|\leqslant\alpha$. (Such a point w_i'' must exist since $\alpha\geqslant\sqrt{d}$.)

(ii) If $a_1+\gamma\geqslant(i+1)\beta$, we define $w_i'=v_i$ and w_i'' as in case (i).

Let F_i be the event that w_i' and w_i'' are open (noting that w_i' is always open under case (ii) above), and that $|C_{w_i''}^\alpha(\omega\cap P_{(i+1)\beta})|<\infty$, so that, by Lemma (L) (4.4.5),

$$\mathbb{P}_p(F_i)\leqslant p\,\mathbb{P}_p(|C_0^\alpha(\omega\cap P_0)|<\infty)\leqslant\gamma(1-p)^{N\alpha^{d-1}}$$

If $C_0^\alpha\cap Q_{n+1}\neq\varnothing$ and $|C_0^\alpha|<\infty$, then all of the events $F_0, F_1,\ldots,F_{r-1}$ have occurred, for r satisfying $r\beta\leqslant n$. It follows as in ref. 11 that

$$\mathbb{P}_p(C_0^\alpha\cap Q_{n+1}\neq\varnothing,\ |C_0^\alpha|<\infty)\leqslant[\gamma(1-p)^{N\alpha^{d-1}}]^{\lfloor n/\beta\rfloor}$$

This implies the claim, since A_n has $2d$ bounding hyperplanes.

Next we indicate how to obtain part (a) of Lemma (L) (4.4.2) when $d\geqslant 3$. Let $\mathscr{L}_r=\{x\in\mathscr{L} : 0\leqslant x_1\leqslant r\}$, a slab of thickness r. We join any two points of $\mathscr{L}_r$ which are separated by distance α or less, and we define the corresponding percolation probability $\theta_r^\alpha(p)=\mathbb{P}_p(|C_0^\alpha\cap\mathscr{L}_r|=\infty)$. There is a "slab" critical probability given by $p_c(\alpha,r)=\sup\{p:\theta_r^\alpha(p)=0\}$. The argument of refs. 12, 13 is easily adapted to obtain that $p_c(\alpha,r)\downarrow p_c(\alpha)$ as $r\to\infty$.

Suppose that $p > p_c(\alpha)$, and find r such that $p > p_c(\alpha, r)$. We now reproduce the essence of the argument in ref. 11, pp. 127–129, adapted as above, obtaining thereby that

$$\mathbb{P}_p(C_0^\alpha \cap Q_{n+1} \neq \varnothing, |C_0^\alpha| < \infty) \leqslant (1 - \theta_r^\alpha(p))^{\lfloor n/(r+\alpha) \rfloor}$$

This implies that

$$\mathbb{P}_p(n < \mathrm{rad}(C_0^\alpha) < \infty) \leqslant 2d \exp(-\theta_r^\alpha(p) \lfloor n/(r+\alpha) \rfloor) \tag{4.4.6}$$

It may be shown in the usual way (see ref. 11, p. 117) that $\theta_r^\alpha(p)$ is a continuous, monotone, and strictly positive function of p on the interval $(p_c(\alpha, r), 1]$. The required conclusion follows easily from the fact that $p_c(\alpha, r) \downarrow p_c(\alpha)$ as $r \to \infty$.

For the remainder of this proof we suppose that $d = 2$. We utilise the block construction of ref. 13 in order to prove Lemma (L) (4.4.2)(a). Suppose $\alpha > 0$ and $p > p_c(\alpha)$, and define the rectangle $T_{M,N} = [0, M] \times [0, N]$ of $\mathbb{R}^2$. A *left-right crossing* of $T_{M,N}$ is a sequence $x_0, x_1, x_2, \dots, x_r$ of points in $\mathscr{L}$ such that

(i) x_i is open and $\|x_{i+1} - x_i\| \leqslant \alpha$ for all i,

(ii) $x_1, x_2, \dots, x_{r-1} \in T_{M,N}$,

(iii) $x_0 \in [-\alpha, 0] \times [0, N]$ and $x_n \in [M, M+\alpha] \times [0, N]$.

A *top-bottom crossing* of $T_{M,N}$ is a sequence satisfying (iii′) in place of (iii), where:

(iii′) $x_0 \in [0, M] \times [N, N+\alpha]$ and $x_n \in [0, M] \times [-\alpha, 0]$.

Before proceeding, we note a geometrical fact. If a rectangle has both a top-bottom and left-right crossing, then the vertices therein belong to the same open cluster of $\mathscr{L}^\alpha$. This follows by use of the triangle inequality.

Let $\mathrm{LR}_{M,N}$ (resp. $\mathrm{TB}_{M,N}$) be the event that $T_{M,N}$ has a left-right (resp. top-bottom) crossing. We claim that, if $\varepsilon > 0$, there exists a positive integer $N = N(p, \alpha, \varepsilon)$ such that

$$\mathbb{P}_p(\mathrm{LR}_{4N,N}) = \mathbb{P}_p(\mathrm{TB}_{N,4N}) > 1 - \varepsilon. \tag{4.4.7}$$

Rather than prove this in detail, we sketch the required argument. The first step is to note that the "block construction" of ref. 13 is valid *mutatis mutandis* for site percolation on $\mathscr{L}_\alpha$. As in ref. 13, one may construct a certain process defined on blocks of $\mathscr{L}$ in such a way that the "block variables" dominate (stochastically) a site percolation process having large density. Since supercritical site percolation (with large density) possesses left-right crossings of tubes with aspect ratio 4, with probability approaching

1 as the size of the tube increases, one obtains (4.4.7) for large N. (The choice of the quantity $4N$ in (4.4.7) is somewhat arbitrary, and may be weakened.)

With N chosen according to (4.4.7), let $R=2N$, and call the box $B_R=[-R, R]^2$ *blue* if

- the two rectangles $[-R, R]\times[-R, -R+N]$ and $[-R, R]\times[R-N, R]$ contain left-right crossings, and
- the two rectangles $[-R, -R+N]\times[-R, R]$ and $[R-N, R]\times[-R, R]$ contain top-bottom crossings.

We extend this definition to all translates of B_R, using the "translated" notions of left-right and top-bottom crossings.

We now renormalise. For $x=(x_1, x_2)\in\mathbb{Z}^2$, we colour x blue if the box $B_R^x=B_R+(3Nx_1, 3Nx_2)$ is blue. We have by (4.4.7) that

$$\mathbb{P}_p(x \text{ is blue})>1-4\varepsilon \tag{4.4.8}$$

and furthermore that the random variables $F=(1\{x \text{ is blue}\} : x\in\mathbb{Z}^2)$ are k-dependent for some value of k which is constant for all α, p, ε, N. Let $p_c(\mathbb{L}^2, \text{site})<\pi<1$, where $p_c(\mathbb{L}^2, \text{site})$ is the critical probability of site percolation on the square lattice $\mathbb{L}^2$. Using the results of ref. 20, we may choose ε, sufficiently small and positive, such that the family F stochastically dominates a site percolation process on $\mathbb{L}^2$ with density π; we choose ε accordingly.

Next we observe a certain property of site percolation at density π, and then we interpret this property in the context of the original process. Consider site percolation on $\mathbb{L}^2$ at density π. Let C_n be the event that the annulus $B_{2n}\backslash B_n$ contains an open circuit D having the origin in its interior, and such that D lies in an infinite open cluster. Using standard arguments (see refs. 11, 17), we have that there exists a function $\rho=\rho(\pi)$, strictly positive and continuous when $\pi>p_c(\mathbb{L}^2, \text{site})$, such that

$$P_\pi(C_n)\geqslant 1-e^{-\rho n} \quad \text{for all large } n \tag{4.4.9}$$

[Here, P_π denotes the appropriate probability measure. Equation (4.4.9) is proved using duality arguments, as follows. If C_n does not occur then either (a) there is a closed matching crossing of the annulus, or (b) there is an open circuit of the annulus which is not connected to infinity. Each eventuality has an exponentially small probability, obtained by applying ref. 11, Theorem (3.4), to the subcritical process on the matching lattice.]

It follows by (4.4.9), and the above remarks concerning stochastic domination, that there exists (with probability at least $1-e^{-\rho n}$) a blue circuit of $B_{2n}\backslash B_n$ which surrounds the origin and is joined to infinity.

We note next that, if $\|x-y\|=1$, then $B_R^x \cap B_R^y$ is a rectangle of dimensions $4N$ by N. Furthermore, if x and y are blue, then the set of eight (left-right or top-bottom) crossings involved in this assumption belong to the same connected open cluster of $\mathscr{L}^\alpha$. Therefore, if the blue circuit exists as in the above paragraph, then either $|C_0^\alpha| = \infty$ or $\mathrm{rad}(C_0^\alpha) \leqslant 6Nn$. We deduce that

$$\mathbb{P}_p(6Nn < \mathrm{rad}(C_0^\alpha) < \infty) \leqslant e^{-\rho n}$$

as required in part (a). That ξ may be assumed continuous in p follows from the fact that N may be assumed bounded away from ∞ when p is bounded away from p_c.

For part (b) of Lemma (L) (4.4.2) when $d=2$, let $0<p_0<1$, $\alpha>\sqrt{20}$, and let $r=\lfloor \alpha/\sqrt{20} \rfloor$, as in the proof of Lemma (L) (4.4.5). With this choice of r, let B^α be the set of all r-occupied points x of $\mathbb{Z}^2$ such that $C_0^\alpha \cap A_r^x \neq \varnothing$. The set B^α forms a collection $B_1, B_2, \ldots$ of connected subsets of vertices of the square lattice, and we write $\Delta_e B_i$ for the external boundary of B_i (i.e., $\Delta_e B_i$ contains all points $y \in \mathbb{Z}^2$ such that (i) y is adjacent in $\mathbb{L}^2$ to some $x \in B_i$, and (ii) y lies in an infinite path of $\mathbb{L}^2$ with points in $\mathbb{Z}^2 \backslash B_i$). We make two claims for the union $\Delta_e B = \Delta_e B_1 \cup \Delta_e B_2 \cup \ldots$, namely:

(A) $\Delta_e B$ is a connected set of points in the matching lattice $\mathbb{L}_*^2$, obtained from $\mathbb{L}^2$ by adding the two diagonals to every unit face, and

(B) for all $y \in \Delta_e B$, the point y is not r-occupied.

Claim (A) follows from consideration of the set of points x for which there exist $u \in A_r^0$, $v \in A_r^x$ satisfying $\|u-v\| \leqslant \alpha$. Claim (B) follows from the fact that, if $\|x-y\|=1$ and $u \in A_r^x$, $v \in A_r^y$, then $\|u-v\| \leqslant \alpha$.

Suppose now that $n < \mathrm{rad}(C_0^\alpha) < \infty$, so that 0 lies in some finite cluster which intersects the complement of A_n. If C_0^α intersects $Q_n = \{x \in \mathbb{R}^d : x_1 \geqslant n\}$, then $\Delta_e B$ contains a path, having an endpoint of the form $(-u, 0)$ with $u \in \{1, 2, \ldots\}$, and with at least $u + n/(2r)$ points in all, none of which is r-occupied. By counting self-avoiding paths on $\mathbb{L}_*^2$, we obtain that

$$\mathbb{P}_p(n < \mathrm{rad}(C_0^\alpha) < \infty) \leqslant 2d \sum_{u=1}^{\infty} \sum_{k \geqslant u + n/(2r)} \{8(1-p)^{4r^2}\}^k.$$

[We have used the fact that $|\mathscr{L} \cap A_r| \geqslant |A_r| = 4r^2$.] We pick a_0 $(>\sqrt{20})$ such that $8^{1/(2r_0)}(1-p_0)^{2r_0} < 1$ where $r_0 = \lfloor a_0/\sqrt{20} \rfloor$, and the claim follows. ∎

5. PROOF OF THEOREM (LP) (3.6), DIFFERENTIABILITY OF κ

In this section we present a summary of the proof that, for given $\alpha > 0$, the function $\kappa_p(\alpha)$ is continuously differentiable in p on the interior of its domain. This will be proved by the method laid out in ref. 11, pp. 78–80, where bond percolation on the cubic lattice was studied.

A separate argument is required in order to show that one-sided derivatives exist at the endpoints of the domain. We omit this, which follows roughly the method used in ref. 11, pp. 140–142.

5.1. Proof of Theorem (LP) (3.6) in the Lattice Case

Let $\mathscr{L}^\alpha$ denote the graph with vertex set $\mathscr{L}$ and with edges joining every pair of vertices which are separated by distance α or less. We define a *distance-α lattice animal* A to be the vertex set of a finite subgraph of $\mathscr{L}^\alpha$ containing the origin. If A is such an animal, we define its boundary ΔA to be the set of sites x in $\mathscr{L}$ for which $0 < \|x, A\| \leqslant \alpha$. Let $\mathscr{A}^\alpha_{nb}$ be the collection of distance-α lattice animals A for which $|A| = n$ and $|\Delta A| = b$, and write $a^\alpha_{nb} = |\mathscr{A}^\alpha_{nb}|$.

There exists a constant $\delta(\mathscr{L})$ such that $|\mathscr{L} \cap B(x, \alpha)| \leqslant \delta(\mathscr{L})\, \alpha^d$ for all $x \in \mathbb{R}^d$ and $\alpha \geqslant \alpha_1$. Therefore

$$1 \leqslant b \leqslant \delta(\mathscr{L})\, \alpha^d n \qquad \text{if} \quad a^\alpha_{nb} \neq 0. \tag{5.1.1}$$

This estimate replaces (4.14)–(4.15) of ref. 11, p. 75.

In order to prove continuous differentiability, it suffices to prove the same property of the quantity

$$p\kappa_p(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_b a^\alpha_{nb} p^n q^b \qquad \text{where} \quad q = 1 - p. \tag{5.1.2}$$

The idea is to show that the sum of the term-wise derivatives in (5.1.2) is uniformly absolutely convergent for $p \in [p_0, p_1]$, whenever $0 < p_0 < p_1 < 1$. The absolute value of the nth term of the sum of derivatives is bounded above by

$$\frac{1}{n} \sum_b a^\alpha_{nb} \left| \frac{n}{p} - \frac{b}{q} \right| p^n q^b.$$

By means of a large-deviations estimate obtained as in ref. 11, Theorem (4.20), one finds that, for any choice of $\xi > 0$, this quantity is no greater than

$$\xi \mathbb{P}_p(|C^\alpha_0| = n) + \frac{\{1 \vee \delta(\mathscr{L})\, \alpha^d\}^2}{pq} n^2 \exp(-\tfrac{1}{3} n \xi^2 p^2 q) \tag{5.1.3}$$

where $u \vee v = \max\{u, v\}$. The proof is completed as in ref. 11, p. 79, by a suitable choice of ξ.

In order to prove the infinite differentiability of κ, one follows the arguments of ref. 11, pp. 140–142, and utilises Lemma (L) (4.4.1), Lemma (L) (4.4.2), and (4.4.3). The only difference of note lies in the subcritical case $p < p_c(\alpha)$, where ref. 11 used an exponential estimate for the *volume* of C_0^α. In our case, Lemma (L) (4.4.1) suffices for the infinite differentiability of $\kappa_p(\alpha)$, via (4.4.3), but is not strong enough to imply real-analyticity. The required arguments are straightforward, and we omit the details. ∎

In the forthcoming proof of Theorem (LP) (3.10), we shall make use of the following result, obtained by summing estimate (5.1.3).

(5.1.4) Corollary (L). Let $0 < p_0 < p_1 < 1$ and $a_0 > 0$. Then $(\partial/\partial p)\,\kappa_p(\alpha)$ is uniformly bounded for $p_0 \leqslant p \leqslant p_1$ and $0 \leqslant \alpha \leqslant a_0$.

5.2. Proof of Theorem (LP) (3.6) in the Poisson Case

We begin with two lemmas, the first of which demonstrates explicitly the dependence on the parameter p of the cluster-size distribution.

(5.2.1) Lemma (P). For each $n \geqslant 1$, there exists a measure $\nu_n = \nu_n^\alpha$ which is concentrated on the interval $[0, (n+1)\,|B(\alpha)|\,]$ and has the property that

$$\mathbb{P}_p(|C_0^\alpha| = n+1 \mid 0 \in \omega) = p^n \int_{(0,\,\infty)} e^{-pV}\, d\nu_n(V) \qquad \text{for} \quad p > 0.$$

Proof. Fix $\alpha \geqslant 0$ and $n \geqslant 1$. Let $\mathbf{x} = \{x_1, x_2, ..., x_n\} \subseteq \mathbb{R}^d$. We write $U_n(\mathbf{x})$ for the indicator function that the graph $G^\alpha(0, \mathbf{x})$, obtained from the vertex set $\{0, x_1, x_2, ..., x_n\}$ by joining all pairs of points which are separated by a distance α or less, is connected. We write $V(\mathbf{x})$ for the volume of the union $D(\mathbf{x}) = B(\alpha) \cup (\bigcup_{m=1}^n B(x_i, \alpha))$ of balls. Now, $|C_0^\alpha| = n+1$ if and only if the following two statements hold: (i) there exists a subset $\mathbf{X} = \{X_1, X_2, ..., X_n\}$ of $\omega \backslash \{0\}$ such that $U_n(\mathbf{X}) = 1$, and (ii) the region $D(\mathbf{X}) \backslash \{0, X_1, X_2, ..., X_n\}$ is empty. Therefore,

$$\mathbb{P}_p(|C_0^\alpha| = n+1 \mid 0 \in \omega) = \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} U_n(\mathbf{x})\, e^{-pV(\mathbf{x})} p^n\, dx_1\, dx_2 \cdots dx_n$$

It follows by a change of variables that

$$\mathbb{P}_p(|C_0^\alpha| = n+1 \mid 0 \in \omega) = p^n \int_{(0,\infty)} e^{-pV}\, d\nu_n(V)$$

for some measure ν_n. Since $|V(\mathbf{x})| \leqslant (n+1) \mid B(\alpha)|$, the measure ν_n is concentrated on the required interval. ∎

The mean number of points within a region of volume V equals pV. The following lemma provides large-deviation bounds for the number of points in a given region.

(5.2.2) Lemma (P). Let ν_n be given as in Lemma (P) (5.2.1). There exists an absolute constant γ_0 such that, for $p, \xi > 0$ and $n \geqslant 0$,

$$\int_{|n-pV|>pn\xi} p^n e^{-pV}\, d\nu_n(V) \leqslant \gamma_0 \sqrt{n} \left\{ [e^{p\xi}(1-p\xi)]^n + [e^{-p\xi}(1+p\xi)]^n \right\}$$

if $p\xi < 1$ and $n > (p\xi)^{-1}$.

Proof. For any $p > 0$

$$\sum_{n=0}^{\infty} \int p^n e^{-pV}\, d\nu_n(V) = \mathbb{P}_p(|C_0^\alpha| < \infty \mid 0 \in \omega) \leqslant 1,$$

whence no term in the sum exceeds 1. Therefore, if $p > 0$ and $0 < v < (n+1)\,|B(\alpha)|$,

$$p^{-n} \geqslant \int e^{-pV}\, d\nu_n(V) \geqslant e^{-pv} \nu_n(0, v),$$

where $\nu_n(a, b)$ is the ν_n-measure of the interval (a, b). Optimising over p for fixed v gives

$$\nu_n(0, v) \leqslant \left(\frac{ev}{n}\right)^n \tag{5.2.3}$$

Split the integral on the left hand side of the inequality in the statement of Lemma (P) (5.2.2) into two pieces corresponding to $V < (n/p)(1-p\xi)$ and $V > (n/p)(1+p\xi)$. In the first integral, integrate by parts and use (5.2.3) to obtain that it equals

$$p^{n+1}\int_{(0,\infty)} e^{-pv}v_n\left(0, v \wedge \left[\frac{n}{p}(1-p\xi)\right]\right) dv$$

$$\leqslant \int_0^{n(1-p\xi)/p} e^{-pv}\left(\frac{epv}{n}\right)^n p\, dv + \int_{n(1-p\xi)/p}^{\infty} e^{-pv}p[e(1-p\xi)]^n\, dv$$

$$= (e/n)^n \int_0^{n(1-p\xi)} u^n e^{-u}\, du + [e^{p\xi}(1-p\xi)]^n$$

Similarly, the second term is bounded by $(e/n)^n \int_{n(1+p\xi)}^{\infty} u^n e^{-u}\, du$. By a refinement of Stirling's formula (see ref. 10, p. 54), $(e/n)^n \leqslant e\sqrt{2\pi n}/n!$, whence the sum of the two terms is bounded above by

$$[e^{p\xi}(1-p\xi)]^n + e\sqrt{2\pi n}\ \mathrm{Prob}\left(\left|\frac{S_{n+1}}{n} - 1\right| > p\xi\right)$$

where S_{n+1} is the sum of $n+1$ independent mean-1 exponential random variables. Using Markov's inequality in the usual way (see ref. 14, p. 184),

$$\mathrm{Prob}(S_n/n > a) \leqslant \exp\{n(1-a+\log a)\} \qquad \text{if} \quad a > 1$$

$$\mathrm{Prob}(S_n/n < a) \leqslant \exp\{n(1-a+\log a)\} \qquad \text{if} \quad 0 < a < 1.$$

Substituting $a = a_n = (1 \pm p\xi)\, n/(n+1)$, we obtain the claim of the lemma. ∎

From (3.1) and Lemma (P) (5.2.1), we have that

$$\kappa_p(\alpha) = \sum_{n=0}^{\infty} \frac{1}{n+1} \int p^n e^{-pV}\, dv_n(V)$$

As in the lattice case, it suffices to show that, for $0 < p_1 < \infty$, the sum of absolute values of the term-wise derivatives converges uniformly for $p \in [p_1, \infty)$. Let $\xi > 0$. The absolute value of the nth term of this sum satisfies

$$\text{(5.2.4)} \qquad \frac{1}{n+1}\left|\int \left(\frac{n}{p} - V\right) p^n e^{-pV}\, dv_n(V)\right|$$

$$\leqslant \int \left|\frac{1}{p} - \frac{V}{n}\right| p^n e^{-pV}\, dv_n(V)$$

$$\leqslant \xi \mathbb{P}_p(|C_0^\alpha| = n+1 \mid 0 \in \omega)$$

$$+ \frac{1 + 2\,|B(\alpha)|}{p} \int_{|n-pV| > pn\xi} p^n e^{-pV}\, dv_n(V)$$

since v_n is supported on $[0, (n+1)\,|B(\alpha)|]$. Choose ξ by $p\xi = (4n^{-1} \log n)^{1/2}$ so that, for large n, $[e^{p\xi}(1-p\xi)]^n \leqslant n^{-2}$ and $[e^{-p\xi}(1+p\xi)]^n \leqslant en^{-2}$. Substituting from Lemma (P) (5.2.2), we obtain that there exists a constant γ_1, depending only on α, such that the right side of (5.2.4) is bounded above by

$$\frac{1}{p}\sqrt{\frac{4\log n}{n}}\,\mathbb{P}_p(|C_0^\alpha| = n+1 \mid 0 \in \omega) + \frac{\gamma_1}{pn^{3/2}}$$

Summing, and using the fact that $\sum_{n=0}^{\infty} \mathbb{P}_p(|C_0^\alpha| = n+1 \mid 0 \in \omega) \leqslant 1$ we obtain that, for N sufficiently large and $p \geqslant p_1$,

$$\sum_{n=N}^{\infty} \left| \frac{d}{dp} \frac{1}{n+1} \int p^n e^{-pV}\, dv_n(V) \right| \leqslant \frac{1}{p_1}\sqrt{\frac{4\log N}{N}} + \frac{\gamma_1}{p_1} \sum_{n=N}^{\infty} \frac{1}{n^{3/2}}$$

The result follows. ∎

6. PROOF OF THEOREM (LP) (3.7), FURTHER PROPERTIES OF H_p

6.1. Proof of Theorem (LP) (3.7)

Suppose that $\omega \in \Omega$ and $0 \in \omega$. The map $\alpha \mapsto |C_0^\alpha|^{-1}$ is readily seen to be right-continuous, and it follows by the bounded convergence theorem that $H_p(\cdot)$ is right-continuous also. Obviously $H_p(0) = 0$. Furthermore,

$$\kappa_p(\alpha) \leqslant N^{-1} + \mathbb{P}_p(|\omega \cap B(\alpha)| \leqslant N-1) \qquad \text{for} \quad N \geqslant 2$$

It follows that $H_p(\alpha) \to 1$ as $\alpha \to \infty$, implying that H_p is a distribution function.

In the Poisson case, (3.2) and Theorem (LP) (3.6) imply that $H_p(\alpha)$ is differentiable in α on $(0, \infty)$. Inequality (3.8) then follows from (4.3.3) and the fact that $1\{V_n - 1 < (1-\varepsilon)\, p\, |A_n|\} \to 0$ a.s. as $n \to \infty$, for fixed $\varepsilon > 0$.

6.2. Further Properties of H_p

In proving part (iii) of Theorem (LP) (3.10) in the lattice case, we shall need to understand the behaviour of the distribution function H_p in the limit as $p \downarrow 0$. We endow the space of probability measures on $\mathbb{R}^+$ with the topology of weak convergence. It is an immediate consequence of Theorem (LP) (3.6) and Theorem (LP) (3.7) that the mapping $p \mapsto H_p$ is a continuous function from $(0, \infty)$ to the space of probability measures on $\mathbb{R}^+$. Furthermore, H_p converges vaguely to H_0 as $p \to 0$; note that H_0 is identically zero, which is not a proper distribution function. In order to control the escape of mass to infinity, we re-scale H_p by defining the function

$$\tilde{H}_p(\alpha) = H_p(p^{-1/d}\alpha) \tag{6.2.1}$$

We shall now state and prove the relevant weak convergence theorem.

(6.2.2) Theorem (LP). We have that $\tilde{H}_p$ converges weakly to H_1^{P} as $p \to 0$.

Proof. This is a trivial consequence of re-scaling in the Poisson case, and therefore we consider only the lattice case. We need to show that, for fixed $\alpha > 0$, $\kappa_p(p^{-1/d}\alpha) \to \kappa_1^{\mathrm{P}}(\alpha)$ as $p \to 0$. Let Ω' be the collection of locally finite subsets ω of $\mathbb{R}^d$ for which $0 \in \omega$; for $\omega \in \Omega'$, define $f_\alpha(\omega) = |C_0^\alpha(\omega)|^{-1}$. Then it suffices to show that

$$\int f_\alpha(\omega)\, d\nu_p(\omega) \to \int f_\alpha(\omega)\, d\nu(\omega) \qquad \text{as} \quad p \to 0 \tag{6.2.3}$$

where ν_p and ν are, respectively, the measures on Ω' corresponding to the set of open sites of site percolation at density p on $p^{1/d}\mathscr{L}$, and the Poisson point process with intensity measure $|\cdot|$, both conditioned to include the origin. Since f_α is bounded, (6.2.3) will follow if we can find a topology on Ω' under which (a) Ω' is a complete separable metric space, (b) ν_p converges weakly to ν, and (c) the set of discontinuities of f_α has measure 0 under ν. A suitable topology is the Skorohod topology. This topology can be specified by requiring that $\omega_n \to \omega$ if and only if $|\omega_n \cap B| \to |\omega \cap B|$ for every open ball $B \subseteq \mathbb{R}^d$. It is easy to check that, under the Skorohod topology, f_α is continuous off the set $\{\omega \in \Omega' : \|x - y\| = \alpha \text{ for some } x, y \in \omega\}$, and that this set has measure 0 under ν. ∎

7. PROOF OF THEOREM (LP) (3.10)(i), CONVERGENCE OF MOMENTS

We give the details only for the lattice case with $\mathscr{L} = \mathbb{Z}^d$. For the other cases, one first re-scales as in Section 4.1, and then uses Lemma (LP) (4.1.1) before proceeding as in the case dealt with here.

We begin by stating a lemma which is an easy consequence of standard theorems (see Theorem 7.10.3 of ref. 14).

(7.1) Lemma. Suppose μ is a positive measure on the measure space $(\Omega, \mathscr{F})$ and f_n, f, g_n, and g are functions which are non-negative, $\mathscr{F}$-measurable, and integrable with respect to μ. Suppose $g_n \to g$ almost everywhere with respect to μ, and $\int g_n \, d\mu \to \int g \, d\mu$ as $n \to \infty$. Then

(i) $\int |g_n - g| \, d\mu \to 0$, and

(ii) if $f_n \leqslant g_n$ and $f_n \to f$ μ-a.e., then $\int |f_n - f| \, d\mu \to 0$.

Since $M_{j,n} \geqslant 0$, it suffices by Lemma (7.1)(i) to show that $M_{j,n} \to m_j(p)$ a.s. and that $\mathbb{E}_p(M_{j,n}) \to m_j(p)$.

Almost Sure Convergence. For $j \geqslant 1$, let λ_j be the measure on $(0, \infty)$ defined by $d\lambda_j(\alpha) = j\alpha^{j-1} \, d\alpha$. We must show that, for almost all ω,

$$\int (1 - F_n(\alpha, \omega)) \, d\lambda_j(\alpha) \to \int \kappa_p(\alpha) \, d\lambda_j(\alpha) \qquad \text{as} \quad n \to \infty$$

It follows from Theorem (LP) (3.3) that, for almost every ω, for every rational α,

$$1 - F_n(\alpha, \omega) \to \kappa_p(\alpha), \qquad \text{and} \qquad 1 - \tilde{F}_n(\alpha, \omega) \to \mathbb{P}_p(X_1 > \alpha)$$

where $\tilde{F}_n$ and X_1 are given as in Theorem (L) (4.2.1). Using Theorem (L) (4.2.1) and Lemma (7.1)(ii), it suffices to show that

$$\int (1 - \tilde{F}_n(\alpha, \omega)) \, d\lambda_j(\alpha) \to \int \mathbb{P}_p(X_1 > \alpha) \, d\lambda_j(\alpha) \quad \text{a.s.}$$

Now $V_n \to \infty$ a.s., and

$$\int (1 - \tilde{F}_n(\alpha, \omega)) \, d\lambda_j(\alpha) = \frac{1}{V_n - 1} \sum_{k=1}^{V_n - 1} X_k(\omega)^j \qquad \text{for} \quad j \geqslant 1$$

By the strong law, both sides converge a.s. to the jth moment of the geometric distribution, as required.

Convergence of Expected Values. If $V_n(\omega) \leqslant 1$, then $F_n(\alpha, \omega) = 0$ for all α. Thus we have (integrating by parts and using Tonelli's theorem) that

$$\mathbb{E}_p(M_{j,n}(\omega)) = \int j\alpha^{j-1} \mathbb{E}_p((1 - F_n(\alpha)) \, 1\{V_n > 1\}) \, d\alpha$$

where we have suppressed reference to ω. However, in the notation of Theorem (L) (4.2.1), we have that $\mathbb{E}_p((1-F_n(\alpha))\,1\{V_n>1\})$ is no greater than

$$\text{(7.2)}\qquad \mathbb{E}_p((1-\tilde{F}_n(\alpha))\,1\{V_n>1\})=\mathbb{E}_p\left(\frac{1\{V_n>1\}}{V_n-1}\sum_{k=1}^{V_n-1}1\{X_k>\alpha\}\right)$$

$$\leqslant 3\mathbb{E}_p\left(\frac{1}{V_n+1}\sum_{k=0}^{V_n}1\{X_k>\alpha\}\right)$$

where the X_i are defined as in the proof of Theorem (L) (4.2.1). Note that $V_n=\inf\{k\colon \sum_{i=0}^{k}X_k>|A_n|\}$, whence V_n is a stopping time with respect to the natural filtration $\{\mathscr{F}_k\colon k\geqslant 0\}$ generated by the sequence $\{X_0, X_1, \ldots\}$. Let $Y_k=1\{X_k>\alpha\}$. We have that

$$\left(\sum_{k=0}^{V_n}Y_k\right)^2=\left(\sum_{k=0}^{V_n}(Y_k-\mathbb{E}_pY_k)\right)^2+(V_n+1)^2\,\mathbb{E}_p(Y_0)^2$$

$$+2(V_n+1)\,\mathbb{E}_p(Y_0)\sum_{k=0}^{V_n}(Y_k-\mathbb{E}_pY_k)$$

whence, by Wald's equation (ref. 14, p. 396) and the Cauchy–Schwarz inequality,

$$\mathbb{E}_p\left[\left(\sum_{k=0}^{V_n}Y_k\right)^2\right]\leqslant S_p(n)+\mathbb{E}_p((V_n+1)^2)\,\mathbb{E}_p(Y_0)^2$$

$$+2\mathbb{E}_p(Y_0)\{\mathbb{E}_p((V_n+1)^2)\,S_p(n)\}^{1/2}$$

where, by Exercise (10.2.15) of ref. 14,

$$S_p(n)=\operatorname{var}_p\left[\sum_{k=0}^{V_n}(Y_k-\mathbb{E}_pY_k)\right]=\mathbb{E}_p(V_n+1)\operatorname{var}_p(Y_0)$$

Now, $\mathbb{E}_p(Y_0^2)=\mathbb{E}_p(Y_0)=\mathbb{P}_p(X_0>\alpha)\leqslant\gamma_0(1-p)^\alpha$ where γ_0 is a constant. Also, V_n is the sum of $|A_n|$ independent identically distributed random variables, whence

$$\mathbb{E}_pV_n=|A_n|\,p,\quad \mathbb{E}_pV_n^2\leqslant|A_n|\,p+(|A_n|\,p)^2,\quad \mathbb{E}_p((V_n+1)^{-2})\leqslant 2(|A_n|\,p)^{-2}$$

By (7.2) and the Cauchy–Schwarz inequality again, we have that

$$\mathbb{E}_p((1-F_n(\alpha))\,1\{V_n>1\})\leqslant\gamma_1(1-p)^{\alpha/2}$$

where γ_1 is a constant. Convergence of $\mathbb{E}_p(M_{n,j})$ to $m_j(p)$ now follows by the dominated convergence theorem and Theorem (LP) (3.3).

8. REMAINING PROOFS, DIFFERENTIABILITY OF THE m_j

There remain the proofs of parts (ii)–(iii) of Theorem (LP) (3.10), and the proof of Theorem (L) (3.11).

8.1. The Lattice Case of Theorem (LP) (3.10)(ii)

We begin with the proof of Theorem (LP) (3.10)(ii) in the lattice case, beginning with the statement that $m_j(p)$ is once differentiable. Let $j \geqslant 1$. By (3.1) and integration by parts, we have that

$$m_j(p) = \int_0^\infty j\alpha^{j-1}\kappa_p(\alpha)\, d\alpha. \tag{8.1.1}$$

Using the mean value and dominated convergence theorems, it is enough to show that, for any $j \geqslant 0$ and any p_0, p_1 with $0 < p_0 < p_1 < 1$,

$$\int_0^\infty \alpha^j \sup_{p \in [p_0, p_1]} \left| \frac{\partial}{\partial p} \kappa_p(\alpha) \right| d\alpha < \infty. \tag{8.1.2}$$

By Corollary (L) (5.1.4), the integrand is uniformly bounded for $0 \leqslant \alpha \leqslant a_0$ where a_0 is as in Lemma (L) (4.4.2)(b). To deal with large α, we utilise as follows the proof that κ is infinitely differentiable for nearest neighbour bond percolation (see ref. 11, Theorem (6.120), pp. 140–141). By Lemma (L) (4.4.2)(b) and (4.4.3), there exist positive constants γ_2 and λ such that, for all n,

$$\mathbb{P}_p(|C_0^\alpha| = n) \leqslant \gamma_2(1-p)^{\lambda n^{1/d}\alpha^{1 \vee (d-2)}} \qquad \text{if} \quad \alpha \geqslant a_0 \quad \text{and} \quad p \geqslant p_0 \tag{8.1.3}$$

Therefore, by (5.1.1), if $\alpha \geqslant a_0$ and $p_0 \leqslant p = 1 - q \leqslant p_1$,

$$\begin{aligned}
\left| \frac{\partial}{\partial p} \kappa_p(\alpha) \right| &\leqslant \sum_{n=1}^\infty \sum_b a_{nb}^\alpha \left| \frac{1}{p} - \frac{b/n}{q} \right| p^n q^b \\
&\leqslant \frac{(1 + \delta(\mathscr{L})\,\alpha^d)}{p_0(1-p_1)} \cdot \sum_{n=1}^\infty \mathbb{P}_p(|C_0^\alpha| = n) \\
&\leqslant \gamma_3 \alpha^d \sum_{n=1}^\infty (1-p_0)^{\lambda n^{1/d}\alpha^{1 \vee (d-2)}} \\
&\leqslant \gamma_4 \alpha^d (1-p_0)^{\lambda \alpha^{1 \vee (d-2)}}
\end{aligned}$$

where the γ_i depend only on $\mathscr{L}$, a_0, p_0, and p_1. Inequality (8.1.2) follows.

Let $r \geqslant 1$. In order to prove that $m_{j+1}(p)$ is r times differentiable off the set $\{p_c(\alpha_k): k \geqslant 1\}$, it suffices to show the following: if $K \geqslant 0$ and $p_c(\alpha_{K+1}) < p_0 < p_1 < p_c(\alpha_K)$, then

$$\int_0^\infty \alpha^j \sup_{p \in [p_0, p_1]} \left| \frac{\partial^r}{\partial p^r} \kappa_p(\alpha) \right| d\alpha < \infty. \tag{8.1.4}$$

The derivatives in (8.1.4) exist by Theorem (LP) (3.6). Furthermore, by (5.1.1) and (5.1.2),

$$\begin{aligned} \left| \frac{\partial^r}{\partial p^r} \kappa_p(\alpha) \right| &\leqslant A(1+\alpha)^{dr} \sum_{n=1}^\infty n^{r-1} \mathbb{P}_p(|C_0^\alpha| = n) \\ &= A(1+\alpha)^{dr} \mathbb{E}_p(|C_0^\alpha|^{r-1}; |C_0^\alpha| < \infty) \end{aligned} \tag{8.1.5}$$

for some $A = A(p_0, p_1, r, \mathscr{L})$.

Note for future use that

$$\kappa_p(\alpha) = \begin{cases} 1 & \text{if } \alpha < \alpha_1, \\ \kappa_p(\alpha_k) & \text{if } \alpha_k \leqslant \alpha < \alpha_{k+1} \end{cases} \tag{8.1.6}$$

where $\alpha_0 = 0$, and also that $\alpha_k \to \infty$ as $k \to \infty$.

Let a_0 ($> \alpha_{K+1}$) be as in Lemma (L) (4.4.2)(b), and split the integral in (8.1.4) into three parts, corresponding to the intervals $[0, \alpha_{K+1})$, $[\alpha_{K+1}, a_0)$, and $[a_0, \infty)$. By (8.1.3) and (8.1.5), the third of these three integrals is finite. The right side of (8.1.5) is non-decreasing in p and α, for $\alpha < \alpha_{K+1}$ and $p < p_c(\alpha_K)$, whence the first integral is no greater than

$$A(1+\alpha_{K+1})^{j+dr+1} \mathbb{E}_{p_1}(|C_0^{\alpha_K}|^{r-1}).$$

This is finite, by Lemma (L) (4.4.1).

By (8.1.6), there exist constants B, L such that the middle integral is no greater than

$$B(1+\alpha_L)^{dr} \sum_{k=K+1}^{L} (\alpha_{k+1}^{j+1} - \alpha_k^{j+1}) \sup_{p_0 \leqslant p \leqslant p_1} \mathbb{E}_p(|C_0^{\alpha_k}|^{r-1}; |C_0^{\alpha_k}| < \infty)$$

which is finite by Lemma (L) (4.4.2)(a) and (4.4.3).

8.2. The Poisson Case of Theorem (LP) (3.10)(ii)

It follows from (3.2) and (8.1.1) that

$$(8.2.1)\qquad m_j^{\mathrm{P}}(p)=\int_0^\infty j\alpha^{j-1}\kappa_p^{\mathrm{P}}(\alpha)\,d\alpha$$

$$=\int_0^\infty j\alpha^{j-1}k_1^{\mathrm{P}}(p^{1/d}\alpha)\,d\alpha=p^{-j/d}m_j^{\mathrm{P}}(1).$$

Finally, $m_j^{\mathrm{P}}(1)<\infty$ by (3.8).

8.3. Proof of Theorem (LP) (3.10)(iii)

This is a trivial consequence of re-scaling for the Poisson model, and so we consider only the lattice model. Let $j\geqslant 1$. We are required to prove that

$$\int_{[0,\infty)}\alpha^j\,d\tilde{H}_p(\alpha)\to\int_{[0,\infty)}\alpha^j\,dH_1^{\mathrm{P}}(\alpha)\qquad\text{as}\quad p\to 0$$

where $\tilde{H}_p$ is given by (6.2.1). Since $\tilde{H}_p\Rightarrow H_1^{\mathrm{P}}$ as $p\to 0$ (cf. Theorem (LP) (6.2.2)), it suffices to prove that, for $\varepsilon>0$, there exists M such that

$$(8.3.1)\qquad \int_M^\infty \alpha^j\,d\tilde{H}_p(\alpha)<\varepsilon\qquad\text{for all}\quad p\in(0,\tfrac{1}{2}]$$

Now, by (6.2.1) and integration by parts,

$$(8.3.2)\qquad M\int_M^\infty\alpha^j\,d\tilde{H}_p(\alpha)\leqslant\int_0^\infty\alpha^{j+1}\,d\tilde{H}_p(\alpha)$$

$$=p^{(j+1)/d}\int_0^\infty\beta^{j+1}\,dH_p(\beta)$$

$$=p^{(j+1)/d}\left\{-\lim_{\beta\to\infty}\left[\beta^{j+1}\kappa_p(\beta)\right]\right.$$

$$\left.+\int_0^\infty(j+1)\,\beta^j\kappa_p(\beta)\,d\beta\right\}$$

By Lemma (LP) (4.3.1), with $E_n(\gamma) = E(\mathscr{T}_n^{\gamma}(\omega))$,

$$\begin{aligned}\kappa_p(\beta) &= \lim_{n\to\infty} \mathbb{E}_p\left(\frac{E_n(\infty) - E_n(\beta)}{V_n - 1}\right) \\ &\leqslant \lim_{n\to\infty}\left[\mathbb{P}_p(V_n \leqslant \tfrac{1}{2}\mathbb{E}_p V_n) + \frac{\mathbb{E}_p(E_n(\infty) - E_n(\beta))}{\tfrac{1}{2}\mathbb{E}_p V_n - 1}\right] \\ &\leqslant \gamma_2 p \exp(-\gamma_1 p \beta^d)\end{aligned}$$

for large β and some positive constants γ_i. The required (8.3.1) follows from (8.3.2). ▌

8.4. Proof of Theorem (L) (3.11)

Let $k, j, r \geqslant 1$ and assume that $p_c(\alpha_k) < 1$. We have from (8.1.1) and (8.1.6) that

$$m_j(p) = (\alpha_{k+1}^j - \alpha_k^j)\,\kappa_p(\alpha_k) + \int_I j\alpha^{j-1}\kappa_p(\alpha)\,d\alpha$$

where $I = [0, \infty)\backslash(\alpha_k, \alpha_{k+1})$. By the argument presented in Section 8.1, the integral is infinitely differentiable at the point $p_c(\alpha_k)$. The claim of the theorem is an immediate consequence. ▌

APPENDIX. GRAPH THEORY

A *graph* Γ consists of a set $\mathscr{V}(\Gamma)$ of points called the *vertices* of Γ, and a set $\mathscr{E}(\Gamma)$ of (unordered) pairs of distinct points of $\mathscr{V}(\Gamma)$ called the *edges* of Γ. If $e = \{v, w\} \in \mathscr{E}(\Gamma)$, v and w are called the *endvertices* of e. A *complete* graph is a graph Γ such that $\mathscr{E}(\Gamma) = \{\{x, y\} : x, y \in \mathscr{V}(\Gamma), x \neq y\}$. The graph Γ is *finite* if $\mathscr{V}(\Gamma)$ is finite. A *subgraph* of a graph Γ is a graph whose vertex set is a subset of $\mathscr{V}(\Gamma)$ and whose edge set is a subset of $\mathscr{E}(\Gamma)$. A *spanning subgraph* of a graph Γ is a subgraph of Γ whose vertex set coincides with that of Γ. A *path* π in a graph Γ is a finite sequence $v_1, e_1, v_2, ..., v_{n-1}, e_{n-1}, v_n$ where the v_i are distinct vertices in $\mathscr{V}(\Gamma)$, and $e_i = \{v_i, v_{i+1}\} \in \mathscr{E}(\Gamma)$. In this case v_1 is called the *initial vertex* of π and v_n its *final vertex*. A *circuit* is a path $v_1, e_1, v_2, ..., v_n$ together with the edge $e_n = \{v_n, v_1\}$. A graph Γ is *connected* if for every pair v, w of vertices in Γ there exists a path with initial vertex v and final vertex w. A *component* in a graph Γ is a maximal connected subgraph of Γ. A *tree* is a connected graph containing no circuits. A *forest* is a graph without circuits. A *spanning* *tree* of a connected graph is a spanning subgraph that is a tree. A *weighted graph* is a graph Γ together with an assignment $e \mapsto w(e)$ of non-negative *weights* to its edges. A *minimal spanning tree* (MST) $\mathscr{T}$ of a weighted graph Γ is a spanning tree of Γ for which $\sum_{e \in \mathscr{E}(\mathscr{T})} w(e)$ is minimal. If ω is a locally finite subset of $\mathbb{R}^d$, a *spanning tree* of ω is a spanning tree of the complete graph with vertex set ω.

Let Γ be a finite weighted graph with n edges. The following routine, called "Kruskal's algorithm", is a standard greedy method for finding a MST in Γ.[7] Let $e_1, ..., e_n$ be a fixed ordering of the edges of Γ such that $w(e_i) \leqslant w(e_{i+1})$ for all i. At stage 0 we are given the vertex set of Γ and no edges. We now examine the edges in the given order. At each stage, we add the current edge e_i to the graph obtained so far if and only if this does not create a circuit. The graph obtained thereby, after all edges have been examined, is a MST.

If the $w(e_i)$ are not distinct, then there may be more than one MST. For any given MST $\mathscr{T}$ of Γ, there exists an ordering of the vertices (as above) for which Kruskal's algorithm gives rise to $\mathscr{T}$.

For any finite graph Γ, we let $V(\Gamma)$, $E(\Gamma)$, and $K(\Gamma)$ respectively denote the number of vertices, edges, and connected components of Γ. It is elementary that, if $\mathscr{F}$ is a finite forest, then

$$E(\mathscr{F}) = V(\mathscr{F}) - K(\mathscr{F}). \tag{A.1}$$

For any weighted graph Γ, we denote by Γ^α the spanning subgraph of Γ obtained by deleting all edges whose weight strictly exceeds α. We shall make use of the following inequality.

(A.2) Lemma. Suppose Γ is any finite connected weighted graph and T is a spanning tree of Γ. Then

$$E(T) - E(T^\alpha) \geqslant K(\Gamma^\alpha) - 1 \qquad \text{for all } \alpha, \tag{A.3}$$

with equality for all α if and only if T is a minimal spanning tree.

Proof. We apply (A.1) to T and to T^α. This yields $E(T) - E(T^\alpha) = K(T^\alpha) - 1$, whence (A.3) follows. For the last statement, it suffices to show that any MST obtained via Kruskal's greedy algorithm satisfies (A.3) with equality. Fix an appropriate ordering of the edges of Γ, and construct a MST $\mathscr{T}$ using Kruskal's algorithm. After all edges of weight α or less have been added, but before any edge with weight exceeding α has been considered, the current graph will be $\mathscr{T}^\alpha$. No edge of length α or less will be considered again, and therefore the components of $\mathscr{T}^\alpha$ are precisely those of Γ^α. Since $\mathcal{T}^\alpha$ is a spanning forest, we have by (A.1) that $E(\mathcal{T}^\alpha) = V(\Gamma) - K(\Gamma^\alpha)$. Taken together with the fact that $E(\mathcal{T}) = V(\Gamma) - 1$, one obtains equality in (A.3). Finally, if a spanning tree T satisfies (A.3) with equality for all α, then $E(T^\alpha) = E(\mathcal{T}^\alpha)$ for any MST $\mathcal{T}$ and all α; it follows that T is a MST. ∎

We define the "edge-weight distribution function" F_Γ of a weighted graph Γ by (2.1).

(A.4) Corollary. Let Γ be a finite connected weighted graph.

(a) Any two MSTs in Γ have the same edge-weight distribution function.

(b) If T is a spanning tree and $\mathcal{T}$ is a MST of Γ, then F_T dominates $F_{\mathcal{T}}$ in the sense that

$$1 - F_T(\alpha) \geqslant 1 - F_{\mathcal{T}}(\alpha) \qquad \text{for all} \quad \alpha \in \mathbb{R}.$$

Proof. Both parts follow from Lemma (A.2) and the fact that $E(T) = V(\Gamma) - 1$ for any spanning tree T. ∎

ACKNOWLEDGMENTS

This work was aided by partial financial support from the European Union under contracts CHRX-CT93-0411 and FMRX-CT96-0075A, and from the Engineering and Physical Sciences Research Council under grant GR/L15426. It was done in Cambridge during a sabbatical visit by C.E.B. from the University of Rochester, and a visit of A.L. from the Université Aix-Marseille III.

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